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# Strong asymptotic behavior for extremal polynomials with respect to varying measures on the unit circle ☆

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**Abstract**

We give a Szegő-type theorem for $L^p$-extremal polynomials with respect to varying measures on $|z| = 1$. Also, we present a density theorem and a generalization of the main result to closed rectifiable Jordan curves and to $|z| = 1$ with the possible addition of a finite number of mass points.


## 1. Introduction

Orthonormal polynomials with respect to varying measures were introduced about 25 years ago by Gonchar and López Lagomasino [5] in connection with a systematic study of the convergence properties of interpolating rational functions with free poles to Markov functions. In a more general context, such approximants are called multipoint Padé approximants. In [5], an analogue of the classical Markov theorem in the theory of continued fractions was proved. Meanwhile, a number of research papers have been devoted to the subject of orthonormal polynomials with

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respect to varying measures; surveys can be found in [12,17], and Chapter 6 of [15]. The emphasis in the present paper is on strong asymptotic, which in the unweighted situation is known as Szegő asymptotics for $L^p$-extremal polynomials [4]. Strong asymptotic results for orthonormal polynomials with respect to varying measures have been proved in various degree of generality in [2,3,11,14,17].

The classical theory of strong asymptotics has its most simple and perhaps also its most natural form for orthonormal polynomials with respect to a measure supported on the unit circle $\mathbb{T} = \{|z| = 1\}$ [16, Chapters XI, XII], or [13, Chapter 3]. Similar results for orthonormal polynomials on a real interval are then usually deduced from the results on $\mathbb{T}$ in a second step. In [10] López–Lagomasino presents orthogonal polynomials with respect to varying measures in a such a way that unifies the theory for the cases of measures with bounded and unbounded support. He also proves asymptotic results for orthogonal polynomials respect to (a fixed) measure with unbounded support using orthogonal polynomials respect to varying measures on $\mathbb{T}$.

Also, these polynomials are very important in Potential Theory. The solution of a number of problems in approximation theory can be reduced to finding the equilibrium distribution of a charge on a "conductor" in the presence of certain "external fields". Problems of this type arise, for example, in the theory of convergence of Padé approximants. In [6], Gonchar and Rakhmanov proved a general theorem which characterizes the limit distribution of the zeros of orthogonal polynomials with respect to varying measures. Also, Gonchar and Rakhmanov proved in [7] a general result concerning the exact rate of best rational approximation for a large class of analytic functions. This result was stated in terms of equilibrium distributions in the presence of external fields and the proof is based on the construction of convenient multipoint Padé approximants whose convergence properties in turn reduces to the study of the limit distribution of zeros of sequences of polynomials which satisfy complex orthogonal relations with respect to varying measures.

In this paper, we give a Szegő-type theorem for $L^p$-extremal polynomials with respect to varying measures on $|z| = 1$. This result will be stated in this section and proved in Section 3. Section 2 is devoted to some auxiliary statements. In Section 4, we present a theorem on density of rational functions and finally, in Section 5, we give some generalizations of the main result. We begin introducing some notations.

Let $\mu$ be a finite positive Borel measure on $[0, 2\pi)$ whose support contains an infinite set of points. In the sequel, we consider $\{W_n\}$, $n \in \mathbb{N}$, a sequence of polynomials such that, for each $n \in \mathbb{N}$, $W_n$ has degree $n$ ($\deg W_n = n$), all its zeros $\{w_{n,i} : 1 \leqslant i \leqslant n\}$ lie in $\mathbb{D} = \{z : |z| < 1\}$, and they satisfy

$$\lim_{n\to\infty} \sum_{i=1}^{n} (1 - |w_{n,i}|) = +\infty. \tag{1}$$

We want to study the asymptotic behavior of polynomials that solve the extremal problem

$$\tau_{n,p} = \inf_{Q_n(z)=z^n+\cdots} \left\| \frac{Q_n}{W_n} \right\|_p = \inf_{Q_n \in \Pi_n, Q_n(0)=1} \left\| \frac{Q_n}{W_n^*} \right\|_p, \tag{2}$$

where $\Pi_n$ is the set of polynomials of degree at most $n$, $W_n^*(z) = z^n \overline{W_n(\frac{1}{\bar{z}})}$, and

$$||f||_p = \left\{ \frac{1}{2\pi} \int |f|^p \, d\mu \right\}^{1/p}.$$

Note that the zeros of $W_n^*(z)$ are $\{\frac{1}{\overline{W_{n,i}}}\}_{i=1,\dots,n} \subset \mathbb{E} = \{|z|>1\}$. From now on, $P_{n,p}$ denotes a polynomial such that

$$\left|\left| \frac{P_{n,p}}{W_n} \right|\right|_p = \tau_{n,p}.$$

Let $\mu'$ be the Radon–Nikodym derivate of $\mu$ with respect to the Lebesgue measure. Assume that $\log \mu' \in L^1$; let $D_p(\mu, z)$ denote the corresponding Szegő function; that is,

$$D_p(\mu, z) = \exp\left\{ \frac{1}{2p\pi} \int_0^{2\pi} \frac{\zeta + z}{\zeta - z} \log \mu'(\theta) \, d\theta \right\}, \quad \zeta = e^{i\theta}, z \in \mathbb{D}.$$

Set

$$K_p(\mu, z) = \begin{cases} \dfrac{D_p(\mu, 0)}{D_p(\mu, z)} & \text{if } z \in S_a \cup \mathbb{D}, \\ 0 & \text{if } z \in S_s, \end{cases} \tag{3}$$

where $S_a$ and $S_s$ give a disjoint decomposition of the unit circle such that $\mu'$ and $\mu_s$ live on these sets, respectively. Hereafter, $\mu_s$ denotes the singular part of $\mu$ with respect to the Lebesgue measure. $H^p$, $0<p<1$, is defined as the class of all functions $f$ analytic in $\mathbb{D}$ such that

$$\sup_{0<r<1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.$$

It is well known that $H^p$ can be identified with the closure in $L^p$ of the set of polynomials in $e^{i\theta}$.

The function $D_p(\mu, z)$ satisfies the following properties:

(1) $D_p(\mu, z)$ is analytic in $\mathbb{D}$; more precisely, $D_p(\mu, z) \in H^p$,
(2) $D_p(\mu, z) \neq 0$ in $\mathbb{D}$, and $D_p(\mu, 0) > 0$,
(3) $|D_p(\mu, e^{i\theta})|^p = \mu'(\theta)$ almost everywhere (a.e.) in $[0, 2\pi]$.

$H^p(\mu)$ is defined as the $L^p(\mu)$ closure of the polynomials in $e^{i\theta}$. $L_s^p(\mu) = \{f \in L^p(\mu) : f = 0, \mu' \text{a.e.}\}$ and $L_a^p(\mu) = \{f \in L^p(\mu) : f = 0, \mu_s \text{a.e.}\}$. Similarly, we define $H_s^p(\mu)$ and $H_a^p(\mu)$.

Moreover, if $f \in H^p(\mu)$, then there exist unique functions $\tilde{f}, f_s$ such that

$$f = K_p \tilde{f} + f_s, \quad \tilde{f} \in H^p, \text{ and } f_s \in L_s^p(\mu). \tag{4}$$

A proof of this result can be seen in [1].

The main theorem of this paper which will be proved in Section 3 is the following:

**Theorem 1.** *For* $0 < p < \infty$, *the following statements are equivalent*:

(i) $\mu$ *satisfies the Szegő condition*; *that is*, $\log \mu' \in L^1$.

(ii) *The following limit exists and is positive*

$$\lim_{n\to\infty} \tau_{n,p} > 0.$$

(iii) *There exists a function* $S \in H^p(\mu)$ *with* $||S||_p \neq 0$, *such that*

$$\lim_{n\to\infty} \left|\left| \frac{P^*_{n,p}}{W^*_n} - S \right|\right|_p = 0.$$

(iv) *There exists a function* $T$ *analytic in* $\mathbb{D}$ *such that*

$$\lim_{n\to\infty} \frac{\dfrac{P^*_{n,p}(z)}{W^*_n(z)}}{\left|\left| \dfrac{P^*_{n,p}}{W^*_n} \right|\right|_p} = T(z)$$

*holds uniformly on each compact subset of* $\mathbb{D}$.

*Moreover*, *if* (i) *holds*, *then*

$$\lim_{n\to\infty} \tau_{n,p} = D_p(\mu, 0),$$

*and the functions in* (iii) *and* (iv) *are* $S(z) = K_p(\mu, z)$ *and* $T(z) = \frac{1}{D_p(\mu,z)}$.

## 2. Auxiliary results

Before we can prove the theorems in the following sections, we need to establish several auxiliary results.

Let $K$ be a compact set and $\{\alpha_{n,1}, \dots, \alpha_{n,n}\} \subset \mathbb{C} \backslash K$ be a given set of points. Let $F_n$ be the set of functions of the form

$$\pi_n(z) = \frac{b_{n,0}z^n + b_{n,1}z^{n-1} + \cdots + b_{n,n}}{(z-\alpha_{n,1})(z-\alpha_{n,2})\cdots(z-\alpha_{n,n})}. \tag{5}$$

Let $f$ be a continuous function on $K$. Denote by $r_n(f)$ the best approximation to $f(z)$ on $K$ in the class $F_n$ in the sense of Tchebycheff; that is,

$$||f - r_n(f)|| = \min\{||f - \pi_n|| : \pi_n \in F_n\}$$

with $||\cdot||$ the supremun norm on $K$.

**Theorem 2.** (See Walsh [18, pp. 246–247]). *Let the points* $\alpha_{n,k}$ *satisfy* $|\alpha_{n,k}|>1$. *A necessary and sufficient condition such that*

$$\lim_{n\to\infty} r_n(f)(z)=f(z), \text{ uniformly in } |z|\leqslant 1, \tag{6}$$

*for every such function f analytic in* $\{|z|\leqslant 1\}$ *is that*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(1-\frac{1}{|\alpha_{n,k}|}\right)=+\infty. \tag{7}$$

A result due to Keldysh is also useful in the proofs that follow. This theorem appears in [9]. An extension of Keldysh's theorem is the following.

**Theorem 3.** (See Bello Hernández et al. [1]).[1] *Let* $\{z_i\}_{i=1,\dots,\Lambda}$ *be a set of points in* $\mathbb{D}$, *where* $\Lambda$ *can be finite or infinite. Let* $\beta$ *be a finite positive Borel measure on* $[0,2\pi)$ *satisfying the Szegő condition and* $\{f_n\}\subset H^p(\beta)$ *(from* (4), $f_n=K_p(\beta,.)\tilde{f}_n+f_{n,s}$), $0<p<\infty$, *such that*

(i) $\lim_{n\to\infty}\tilde{f}_n(0)=1$;
(ii) $\lim_{n\to\infty}\tilde{f}_n(z_i)=0,\ i=1,2,\dots$;
(iii) $\sum_{i=1}^{\Lambda}(1-|z_i|)<+\infty$;
(iv) $\lim_{n\to\infty}\left(\int|f_n(e^{i\theta})|^p d\beta(\theta)\right)^{1/p}=\dfrac{D_p(\beta,0)}{\prod_{i=1}^{\Lambda}|z_i|}$.

*Then*

(a) $\lim_{n\to\infty}\tilde{f}_n(z)=\prod_{i=1}^{\Lambda}\dfrac{z-z_i}{\bar{z}_i z-1}\dfrac{\bar{z}_i}{|z_i|^2}$ *holds uniformly on each compact subset of* $\mathbb{D}$, *and*

(b) $\lim_{n\to\infty}\displaystyle\int\left|f_n-K_p(\beta,z)\prod_{i=1}^{\Lambda}\frac{z-z_i}{\bar{z}_i z-1}\frac{\bar{z}_i}{|z_i|^2}\right|^p d\beta=0.$

If $\Lambda$ is an empty set then the right-hand side of (a) is equal to 1; that is, $\prod_{i=1}^{\Lambda}\frac{z-z_i}{\bar{z}_i z-1}\frac{\bar{z}_i}{|z_i|^2}\equiv 1$.

## 3. Proof of Theorem 1

Before proving Theorem 1, we show an intermediate result.

[1] We wish to point out a missprint in condition (iv) of the mentioned reference. It is necessary to substitute $\prod_{i=1}^{\Lambda}|z_i|^p$ by $\prod_{i=1}^{\Lambda}|z_i|$.

**Theorem 4.** *For* $0<p<\infty$

$$\lim_{n\to\infty}\left|\left|\frac{P_{n,p}}{W_n}\right|\right|_p = D_p(\mu,0),$$

*where* 0 *replaces* $D_p(\mu,0)$ *if* $\log\mu'(\theta)$ *is not integrable.*

**Proof.** Let $\Lambda\subset\mathbb{N}$ be an indexed sequence such that

$$\lim_{n\in\Lambda}\left|\left|\frac{P^*_{n,p}}{W^*_n}\right|\right|_p = \limsup_{n\to\infty}\left|\left|\frac{P^*_{n,p}}{W^*_n}\right|\right|_p. \tag{8}$$

From a result due to Szegő (see [16, p. 297]), we know that if $T_{n,2}$ are the extremal polynomials such that

$$||T_{n,2}||_2 = \min\{||Q_n||_2 : Q_n \text{ monic of degree } n\},$$

then

$$\lim_n ||T_{n,2}||_2^2 = \lim_n ||T^*_{n,2}||_2^2 = D_2(\mu,0)^2, \tag{9}$$

with $D_2(\mu,0)=0$ if $\log\mu'(\theta)$ is not integrable. Since the zeros of $T^*_{n,2}$ lie in $\mathbb{E}$, $\varphi_n(z) = (T^*_{n,2}(z))^{2/p}$ is analytic in $\{|z|\leqslant 1\}$, so as (1) holds, from Theorem 2 there exists a sequence $\{\frac{R_{m_n}}{W^*_{m_n}}\}_{m_n\in\Lambda'\subset\Lambda}$ such that

$$\lim_{n\in\Lambda}\sup_{|z|\leqslant 1}\left|\frac{R_{m_n}(z)}{W^*_{m_n}(z)} - \varphi_n(z)\right| = 0.$$

In particular, since $\varphi_n(0) = (T^*_{n,2}(0))^{p/2} = 1 = W^*_{m_n}(0)$, we have $\lim_{n\in\Lambda}R_{m_n}(0)=1$. Hence

$$\lim_{n\in\Lambda}\left|\left|\frac{R_{m_n}}{W^*_{m_n}}\right|\right|_p^p = \lim_{n\in\Lambda}||\varphi_n||_p^p = \lim_{n\in\Lambda}||T^*_{n,2}||_2^2 = D_2(\mu,0)^2 = D_p(\mu,0)^p$$

and

$$\limsup_{n\to\infty}\left|\left|\frac{P^*_{n,p}}{W^*_n}\right|\right|_p \leqslant D_p(\mu,0). \tag{10}$$

On the other hand, using Jensen's inequality

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P^*_{n,p}(e^{i\theta})}{W^*_n(e^{i\theta})}\right|^p \mathrm{d}\mu(\theta) \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P^*_{n,p}(e^{i\theta})}{W^*_n(e^{i\theta})}\right|^p \mu'(\theta)\,\mathrm{d}\theta$$

$$\geqslant \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{P^*_{n,p}(e^{i\theta})}{W^*_n(e^{i\theta})}\right|^p \mathrm{d}\theta\right\}$$

$$\times\ \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\log\mu'(\theta)\,\mathrm{d}\theta\right\}$$

$$\geqslant \left|\frac{P^*_{n,p}(0)}{W^*_n(0)}\right|^p D_p(\mu,0)^p = D_p(\mu,0)^p.$$

Therefore,

$$\liminf_{n\to\infty}\left\|\frac{P^*_{n,p}}{W^*_n}\right\|_p \geqslant D_p(\mu,0). \tag{11}$$

With (10) and (11) the theorem is proved. □

**Remark 1.** We also proved that

$$\lim_{n\to\infty}\left\|\frac{P^*_{n,p}}{W^*_n}\right\|_p = \lim_{n\to\infty}\left(\frac{1}{2\pi}\int\left|\frac{P^*_{n,p}(z)}{W^*_n(z)}\right|^p \mu'(\theta)\,\mathrm{d}\theta\right)^{1/p} = D_p(\mu,0).$$

**Proof of Theorem 1.**

**Proof.** (i)$\Leftrightarrow$ (ii): It follows from Theorem 4.

(i)$\Rightarrow$ (iii): We consider the function

$$h_n(z) = \frac{P^*_{n,p}(z)D_p(\mu,z)}{W^*_n(z)D_p(\mu,0)}, \tag{12}$$

that belongs to $H^p$. Since $h_n(0)=1$ and $|D_p(\mu,e^{i\theta})|^p = \mu'(\theta)$, from Theorem 4, we have

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|h_n(e^{i\theta})|^p\,\mathrm{d}\theta\right\} = 1. \tag{13}$$

Applying Theorem 3 (here $\Lambda=\emptyset$ and $\beta$ is the Lebesgue measure) it follows that

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}|h_n(e^{i\theta})-1|^p\,\mathrm{d}\theta\right\} = 0.$$

Hence

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P^*_{n,p}(e^{i\theta})D_p(\mu,e^{i\theta})}{W^*_n(e^{i\theta})D_p(\mu,0)}-1\right|^p \mathrm{d}\theta\right\} = 0$$

and then

$$\lim_{n\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{P^*_{n,p}(e^{i\theta})}{W^*_n(e^{i\theta})}-\frac{D_p(\mu,0)}{D_p(\mu,e^{i\theta})}\right|^p\mu'(\theta)\,\mathrm{d}\theta\right\}=0.$$

Therefore, using (13) and again Theorem 4, we obtain (iii) where $S(z)=K_p(\mu,z)$.

(iii)⇒ (i): It follows from the relation

$$\liminf_{n\to\infty}\left|\left|\frac{P^*_{n,p}}{W^*_n}-\psi\right|\right|_p=0, \tag{14}$$

where $\psi(\zeta)$ is such that $||\psi||_p\neq 0$.

Indeed, according to (14) we have that there exists a subsequence $\{n_v\}$ such that

$$\lim_{v\to\infty}\left|\left|\frac{P^*_{n_v,p}}{W^*_{n_v}}-\psi\right|\right|_p=0.$$

If (i) does not hold, from Theorem 4

$$\lim_{v\to\infty}\left|\left|\frac{P^*_{n_v,p}}{W^*_{n_v}}\right|\right|_p=0,$$

and we obtain $||\psi||_p=0$, which is a contradiction.

(iii)⇒ (iv): The sequence of functions $\{h_n\}$ as in (12) satisfies the hypothesis of Theorem 3, hence $\lim_{n\to\infty}h_n(z)=1$ holds uniformly on each compact subset of $\mathbb{D}$. Now, since (i) is equivalent to (iii), using again Theorem 4, we obtain

$$\lim_{n\to\infty}\frac{\dfrac{P^*_{n,p}(z)}{W^*_n(z)}}{\left|\left|\dfrac{P^*_{n,p}(z)}{W^*_n(z)}\right|\right|_p}=\frac{1}{D_p(\mu,z)}.$$

(iv)⇒ (i). From (iv) and Theorem 4, we have

$$\lim_{n\to\infty}\frac{\dfrac{P^*_{n,p}(0)}{W^*_n(0)}}{\left|\left|\dfrac{P^*_{n,p}(z)}{W^*_n(z)}\right|\right|_p}=T(0)=\frac{1}{D_p(\mu,0)}<\infty,$$

but this is true if and only if (i) holds. □

## 4. Density Theorem

In this section we give a density theorem that can be seen as an "application" of the main result.

We introduce the notation: $R_{n,k}=\{\frac{h}{W^*_n}:h\in\Pi_{n-k}\}$.

**Theorem 5.** *Assume that $\mu$ is an absolutely continuous measure $\mu_a$ and satisfies Szegő's condition, then the following statements are equivalent*:

(i) *For each* $j \in \mathbb{Z}_+$

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{P^*_{n,n-j,\,p}(e^{i\theta})}{W^*_n(e^{i\theta})} \right|^p \mathrm{d}\mu_a(\theta) = D_p(\mu_a, 0)^p,$$

*where $P_{n,n-j,\,p}$ is the monic extremal polynomial*; *that is*,

$$\left\| \frac{P_{n,n-j,\,p}}{W_n} \right\|_p = \min\left\{ \left\| \frac{Q_{n-j}}{W_n} \right\|_p : Q_{n-j} \in \Pi_{n-j},\ \text{monic} \right\}.$$

(ii) *For each* $k \in \mathbb{Z}_+$, $R_{n,k}$ *is dense in* $H^p(\mu_a)$.

**Proof.** (ii)$\Rightarrow$ (i): Here we use the same technique as in the proof of Theorem 4. It is well known that (ii)$\Leftrightarrow$ (i) is true when $p=2$ and $W_n(z)=z^n$ (see, for example, [16, p. 297]). Let $T_{n-k,2}$ denote the monic extremal polynomials in this case, then given $k \geqslant 0$ there exist polynomials $R_{m_n-k}$ of degree $m_n-k$ such that

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left| (T^*_{n-k}(z))^{2/p} - \frac{R_{m_n-k}(z)}{W^*_{m_n}(z)} \right|^p \mu'_a(\theta)\,\mathrm{d}\theta = 0.$$

Notice that the functions $\{(T^*_{n-k})^{2/p}\}_n$ are analytic in an open set containing $\{|z| \leqslant 1\}$ because $T^*_{n-k}$ has no zeros in $\{|z| \leqslant 1\}$. Then

(a) $\lim_{n\to\infty} R_{m_n-k}(0) = 1$;

(b) $\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{R_{m_n-k}(z)}{W^*_{m_n}(z)} \right|^p \mu'_a(\theta)\,\mathrm{d}\theta = D_p(\mu_a, 0)^p$.

Given $\Lambda \subset \mathbb{N}$ an indexed sequence, from (ii) we observe that the sequence $\{m_n\}$ can be chosen in $\Lambda$. Therefore, (i) follows from (a) and (b).

(i)$\Rightarrow$ (ii): Set $i, j \in \mathbb{Z}_+$, using (i) and Theorem 3, we have

$$\frac{P^*_{n,n-(i+j),\,p}(z)}{W^*_n(z)} \to K_p(\mu_a, z)$$

in $L^p(\mu_a)$. Then

$$\frac{z^i P^*_{n,n-(i+j),\,p}(z)}{W^*_n(z)} \to z^i K_p(\mu_a, z),$$

in $H^p(\mu_a)$. Since $H^p(\mu_a) = H^p \cdot K_p(\mu_a, \cdot)$ and $H^p$ is the closure of the polynomials in $L^p$, $R_{n,\,j}$ satisfies (ii). $\square$

## 5. Generalizations

In this section we give two generalizations of Szegő's theorem for extremal rational functions. First, for a closed rectifiable Jordan curve $C$, and second, for sets of the form $\mathbb{F} = \mathbb{T} \cup \{z_1, z_2, \dots, z_N\}$, with $z_i \in \mathbb{D}$. Szegő's theorem on $\mathbb{F}$ respect to a fixed measure was proved by Kaliaguine [8].

With the same techniques used in Theorem 1, we can prove an analogous theorem for closed rectifiable Jordan curves. We begin introducing the necessary notation. Let $\alpha \in \mathbb{C}$ and let $C$ be a closed rectifiable Jordan curve with length $l$ in the $z$-plane in whose interior lies $\alpha$. Let $\sigma(s)$ be a finite positive measure on $[0, l)$. As usual, by $L^p(C, \sigma)$ we denote the space of complex measurable functions on $C$, such that

$$||f||_{p,\sigma} = \left\{ \frac{1}{2\pi} \int_C |f(\zeta)|^p \, \mathrm{d}\sigma(s) \right\}^{1/p} < \infty, \quad \zeta = \zeta(s), \quad s \in [0, l),$$

with $\zeta = \zeta(s)$ a parametrization of $C$. Let $B$ denote the interior of $C$ and consider the conformal transformation

$$x = \varphi(z) = \alpha + z + b_2 z^2 + \cdots, |z| < 1, \ \alpha \in B$$

which maps $\mathbb{D}$ onto $B$, such that $\alpha = \varphi(0)$ and $\varphi'(0) > 0$. From Caratheodory's theorem $\varphi$ can be extended continuously to $\{|z| \leqslant 1\}$ so that $\varphi$ is a bijection from $\{|z| = 1\}$ to $C$ whose inverse we denote by $z = \gamma(x)$. Then, the measure $\sigma$ induces an image measure $\mu$ on $|z| = 1$ by $\mu(E) = \sigma(\zeta^{-1}(\gamma^{-1}(E)) = \sigma((\gamma \circ \zeta)^{-1}(E))$, thus

$$\sigma'(s)|d\zeta| = \sigma'(s)|\varphi'(e^{i\theta})| \, \mathrm{d}\theta = \mu'(\theta) \, \mathrm{d}\theta.$$

Let $J$ be the unbounded component of the complement of $C$ and $z = \phi(x)$ the conformal transformation which maps $J$ onto $\mathbb{E}$ so that the points at infinity correspond to each other and $\phi'(\infty) > 0$. Let $C_R$ denote generically the curve $|\phi(x)| = R > 1$ in $J$.

We define $d\sigma_n = \frac{d\sigma}{|Y_n|^p}$, where $\{Y_n\}$, $n \in \mathbb{N}$ is a sequence of polynomials such that for each $n$, $Y_n$ has exactly degree $n$, and all its zeros $(\alpha_{n,i})_{i=1,\dots;n \in \mathbb{N}}$ have no limit points interior to $C_A$, $A > 1$ and $Y_n(\alpha) = 1$.

We want to study the asymptotic behavior of the polynomials $\tilde{P}_{n,p}$, that solve the following extremal problem:

$$\rho_{n,p} = \inf_{Q_n(\alpha)=1} ||Q_n||_{p,\sigma_n} = \inf_{Q_n(\alpha)=1} \left\{ \frac{1}{2\pi} \int_C \left| \frac{Q_n(\zeta)}{Y_n(\zeta)} \right|^p d\sigma(s) \right\}^{1/p}. \tag{15}$$

First we quote a result analogous to Theorem 2.

**Theorem 6** (See Walsh [18], pp. 252–253). *Let $C$ be a closed rectifiable Jordan curve and let the points $(\alpha_{n,i})_{i=1,\dots;n \in \mathbb{N}}$ be given with no limit points interior to $C_A$. If $f$ is an analytic function on and within $C_T$, there exists a sequence $r_n$ of functions of the form* (5) *such that*

$$\lim_{n \to \infty} r_n(x) = f(x),$$

*uniformly for $x$ on each closed subset of interior of $C_R$, where $R = \frac{A^2T + T + 2A}{2AT + A^2 + 1}$.*

Using this result we can obtain a theorem analogous to Theorem 1:

**Theorem 7.** *For* $0<p<\infty$, *the following statements are equivalent*:

(i) $\sigma$ *satisfies the Szegő condition.*

(ii) *The following limit exists and is positive*:

$$\lim_{n\to\infty} \rho_{n,p} > 0.$$

(iii) *There exists* $S \in L^p(C, \sigma)$ *with* $||S||_{p,\sigma} \neq 0$, *such that*

$$\lim_{n\to\infty} \left\| \frac{\tilde{P}_{n,p}(x)}{Y_n(x)} - S(x) \right\|_{p,\sigma} = 0.$$

(iv) *There exists a function* $T$ *analytic in* $B$ *such that*

$$\lim_{n\to\infty} \frac{\frac{\tilde{P}_{n,p}(x)}{Y_n(x)}}{\left\| \frac{\tilde{P}_{n,p}}{Y_n} \right\|_{p,\sigma}} = T(x)$$

*holds uniformly on each compact subset of* $B$.

*Moreover, if* (i) *holds*, $\lim_{n\to\infty} \rho_{n,p} = \Delta_p(\sigma, \alpha)$, $S(x) = J_p(\sigma, x)$, and $T(x) = \frac{1}{\Delta_p(\sigma,x)}$, where $\Delta_p(\sigma, x) = D_p(\mu, \gamma(x))$ and $J_p(\sigma, x) = K_p(\mu, \gamma(x))$.

This theorem can also be given for sets of the form $\mathbb{F} = \mathbb{T} \cup \{z_1, z_2, \ldots, z_N\}$ with $z_k \in \mathbb{D}$. Let $\beta_n$ be a varying measure such that $\beta_n = \mu_n + \eta$, where $\eta$ is a discrete measure with mass $A_k > 0$ at the point $z_k$, $k = 1, \ldots, N$, and $d\mu_n = \frac{d\mu}{|W_n^*|^p}$ the varying measure with $\mu$ and $W_n$ as in Section 1.

Here, we study the asymptotic behavior of the polynomials $T_{n,p}^*(z, \beta_n)$, that solve the extremal problem

$$\lambda_{n,p} = \min_{P_n(0)=1} \left\{ \frac{1}{2\pi} \int_{|z|=1} \left| \frac{P_n(z)}{W_n^*(z)} \right|^p d\mu(\theta) + \sum_{k=1}^{N} |P_n(z_k)|^p A_k \right\}^{1/p}. \tag{16}$$

**Theorem 8.** *For* $0<p<\infty$ *the following statements are equivalent*:

(i) $\mu$ *satisfies the Szegő condition.*

(ii) *The following limit exists and is positive*:

$$\lim_{n\to\infty} \lambda_{n,p} > 0.$$

(iii) *There exists a function* $S \in H^p(\mu)$ *with* $||S||_{p,\mu} \neq 0$ *such that*

$$\lim_{n\to\infty} \left|\left| \frac{T^*_{n,p}(z,\beta_n)}{W^*_n(z)} - S(z) \right|\right|_{p,\mu} = 0.$$

(iv) *There exists a function* $T$ *analytic in* $\mathbb{D}$ *such that*

$$\lim_{n\to\infty} \frac{\dfrac{T^*_{n,p}(z,\beta_n)}{W^*_n(z)}}{\left|\left|\dfrac{T^*_{n,p}(.,\beta_n)}{W^*_n}\right|\right|_{p,\mu}} = T(z)$$

*holds uniformly on each compact subset of* $\mathbb{D}$.

*Moreover, if* (i) *holds*

$$\lim_{n\to\infty} \lambda_{n,p} = D_p(\mu,0)$$

*and the functions in* (iii) *and* (iv) *are* $S(z) = \frac{D_p(\mu,0)}{D_p(\mu,z)}$ *and* $T(z) = \frac{1}{D_p(\mu,z)}$, *respectively.*

**Proof.** We will only prove (i)$\Leftrightarrow$ (ii) because the rest of the proof is similar to that given in Theorem 1.

Since $A_k > 0$,

$$\lambda_{n,p} \geqslant \inf_{Q_n(0)=1} \left\{ \frac{1}{2\pi} \int_{|z|=1} \left| \frac{Q_n(z)}{W^*_n(z)} \right|^p d\mu(\theta) \right\}^{1/p} = \tau_{n,p},$$

and from Theorem 1

$$\liminf_{n\to\infty} \lambda_{n,p} \geqslant D_p(\mu,0). \tag{17}$$

Now, let $V_N$ be the polynomial whose zeros are $z_1, z_2, \ldots, z_N$ and let $T^*_{n-N,p}$, with $T^*_{n-N,p}(0) = 1$, be the extremal polynomial for the measure $|\frac{V_N(z)}{V_N(0)}|^p \, d\mu_n$; that is,

$$\lambda^p_{n,p} \leqslant \inf_{Q_{n-N}(0)=1} \int_{|z|=1} \left| \frac{Q_{n-N}(z)}{W^*_n(z)} \right|^p \left| \frac{V_N(z)}{V_N(0)} \right|^p d\mu$$
$$= \int_{|z|=1} \left| \frac{T^*_{n-N,p}(z)}{W^*_n(z)} \right|^p \left| \frac{V_N(z)}{V_N(0)} \right|^p d\mu.$$

Using the Theorem 4, we have

$$\lim_{n\to\infty} \int_{|z|=1} \left| \frac{T^*_{n-N,p}(z)}{W^*_n(z)} \right|^p \left| \frac{V_N(z)}{V_N(0)} \right|^p d\mu = D_p\left( \left| \frac{V_N(z)}{V_N(0)} \right|^p d\mu, 0 \right)^p.$$

From the properties of the Szegő function, we obtain that

$$D_p\left(\left|\frac{V_N(z)}{V_N(0)}\right|^p d\mu, 0\right) = D_p(\mu, 0)$$

and hence

$$\limsup_{n\to\infty} \lambda_{n,p} \leqslant D_p(\mu, 0). \tag{18}$$

The result follows from (17) and (18). □

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